To prove the Mean Value Theorem, we need two other theorems:

**Extreme Value Theorem**

If $f(x)$ is continuous on the interval $[a, b]$ then there are two numbers, $a \leq c$ and $d \leq b$ such that $f(c)$ is a maximum for the function on the interval and $f(d)$ is a minimum for the function on the interval.

Basically this theorem says that, on a closed interval that includes the endpoints, somewhere there will be a maximum and a minimum of the function on the interval.

**Rolle’s Theorem**

Suppose $f(x)$ is a function that satisfies the following conditions:

1. $f(x)$ is continuous on $[a, b]$
2. $f(x)$ is differentiable on $(a, b)$
3. $f(a) = f(b)$

Then there is a number $c$ on $(a, b)$ where $f'(c) = 0$.

The proof of Rolle’s Theorem requires the consideration of three different cases:

**Case 1:** $f(x) = k$ for all $[a, b]$

If the function is a constant for the entire interval then we know the slope is 0 at all $c$ in $[a, b]$.

**Case 2:** There is some number $d$ in $(a, b)$ such that $f(d) > f(a)$.

Because $f(x)$ is continuous on $[a, b]$, by the Extreme Value Theorem, we know that $f(x)$ will have a maximum somewhere on $[a, b]$. Given that $f(a) = f(b)$ and $f(d) > f(a)$ we know that the maximum must occur somewhere on $(a, b)$ (notice the change in the interval). We know the derivative must exist because $f(x)$ is differentiable on $(a, b)$ (see given #2) and that the derivative is zero at the maximum so somewhere on $(a, b)$, $f'(c) = 0$ (where $a < c < b$).

**Case 3:** There is some number $d$ in $(a, b)$ such that $f(d) < f(a)$.

Because $f(x)$ is continuous on $[a, b]$, by the Extreme Value Theorem, we know that $f(x)$ will have a minimum somewhere on $[a, b]$. The rest of the proof of this case is similar to case 2.

We needed the Extreme Value Theorem to prove Rolle’s Theorem. We need Rolle’s Theorem to prove the Mean Value Theorem.

Mean Value Theorem

The equation of the secant line through \((a, f(a))\) and \((b, f(b))\) is

\[ y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a) \]

so we can say that the \(y\)-coordinate on the secant line is

\[ y = f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \]

Let us define the function \(g(x)\) to be the difference between \(y\) and \(f(x)\)

\[ g(x) = f(x) - y = f(x) - \left[ f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right] \]

We know that \(g(a) = g(b) = 0\) (the \(y\)-coordinate on the secant line exactly matches the function at the endpoints of the secant line). Also, \(g\) is continuous on \([a, b]\) and differentiable on \((a, b)\) – known because \(g(x)\) is the difference between \(f(x)\) and a linear polynomial, both of which are continuous and differentiable on \([a, b]\) and \((a, b)\) respectively. This means that \(g(x)\) satisfies the requirements for Rolle’s Theorem so there must be some point \(c\) on the interval \((a, b)\) such that \(g'(c) = 0\).

\[ g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \]

\[ 0 = f'(c) - \frac{f(b) - f(a)}{b - a} \]

\[ f'(c) = \frac{f(b) - f(a)}{b - a} \]