Proof of the Clairaut Theorem (Equality of the Mixed Partial Derivatives)

Suppose \( f(x, y) \) is a function of two variables such that \( f_{xy}(x, y) \) and \( f_{yx}(x, y) \) exist and are continuous. Further suppose that all of the limits used in this proof exist. Then,

\[
f_{xy}(x, y) = \lim_{\Delta y \to 0} \frac{f_x(x, y + \Delta y) - f_x(x, y)}{\Delta y}
= \lim_{\Delta y \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y)}{\Delta y}
= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) - f(x + \Delta x, y) + f(x, y)}{\Delta y \Delta x}
\]

In the same way,

\[
f_{yx}(x, y) = \lim_{\Delta x \to 0} \frac{f_y(x + \Delta x, y) - f_y(x, y)}{\Delta x}
= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y)}{\Delta x}
= \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) - f(x, y + \Delta y) + f(x, y)}{\Delta x \Delta y}
\]

Thus, the limit for \( f_{xy}(x, y) \) is equal to the limit for \( f_{yx}(x, y) \).

Intuition behind the theorem: When taking the mixed partial, we are essentially dividing by \( \Delta x \) and \( \Delta y \). Since the order in which we do this division doesn’t matter, it makes sense that \( f_{xy}(x, y) = f_{yx}(x, y) \).