Answers are in blue unless I had to type in equations. I have tried to show enough work to answer questions. You can email me if necessary.

Extra problems on trigonometric identities

1. **a.** If a line makes an angle of θ with a horizontal line (such as the *x*-axis), θ is called an *angle of inclination* for the line. Suppose that the line has slope *m*. Prove that $m = \tan\theta$. You may need to consider positive and negative slopes as separate cases.

This is of course because slope is rise/run and sin(θ) = rise, cos(θ) = run.

b. Suppose that two lines have slopes m_1 and m_2 and angles of inclination α and β. Note that (α – β) is the measure of an angle formed between the two lines. Use the tan(α – β) identity to get a formula for tan(α – β) in terms of m_1 and m_2.

See pages 376-377 of the Brown book for a complete explanation. $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$.

Think how this is related to the identity $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

Before continuing, check your answer to problem **1b** against the *Adv. Math.* book, page 377 top, because you'll be using this formula to do subsequent problems.

2. Consider the lines $y = 2x$ and $y = 3x$. Find the measure of the angle between these lines, two different ways:

a. Find an angle of inclination for $y = 2x$ and an angle of inclination for $y = 3x$.
Then the angle between the lines is the difference between these angles of inclination.

$\tan^{-1}(3) - \tan^{-1}(2) = 1.249 - 1.107 = 0.142 = 8.13°$

b. Use the formula found in problem **1b**.

$$\theta = \tan^{-1}\left(\frac{3-2}{1+3\cdot 2}\right) = \tan^{-1}\left(\frac{1}{7}\right) = 0.142 = 8.13°$$

3. Show that $\tan(x + \frac{\pi}{4}) = \frac{1+\tan(x)}{1-\tan(x)}$.

$$\tan(x + \tfrac{\pi}{4}) = \frac{\tan(x) + \tan\left(\frac{\pi}{4}\right)}{1 - \tan(x)\tan\left(\frac{\pi}{4}\right)} = \frac{\tan(x)+1}{1-\tan(x)\cdot 1} = \frac{1+\tan(x)}{1-\tan(x)}$$

Thanks to Mr. Kelly for these.

4. Since tan(x) and tan($x + \frac{\pi}{4}$) can be interpreted as slopes of lines (see problem **1**), you can think of the fact proved in problem **3** as giving a relationship between the slopes of two lines whose angles are $\frac{\pi}{4}$ radians apart; in other words, the lines form a $\frac{\pi}{4}$ radian (or 45°) angle. Use this approach in the following problems.

a. Suppose that a line with slope $\frac{1}{5}$ is rotated counterclockwise by 45°. Find the slope of the resulting line. $\tan^{-1}\left(\frac{1}{5}\right) = 0.197 = 11.309°$ is the original angle of the line with the x-axis.

The new line forms an angle of 0.197 + $\frac{\pi}{4}$ so the new slope is $\tan\left(0.197 + \frac{\pi}{4}\right) = 1.5$.

b. Repeat part **a** for a slope of $\frac{1}{n}$. $\tan\left(\tan^{-1}\left(\frac{1}{n}\right) + \frac{\pi}{4}\right) = \dfrac{1 + \tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)}{1 - \tan\left(\tan^{-1}\left(\frac{1}{n}\right)\right)} = \dfrac{1 + \frac{1}{n}}{1 - \frac{1}{n}} = \dfrac{n+1}{n-1}$

c. Show that a pair of lines whose slopes are $\frac{2}{7}$ and $\frac{9}{5}$ form a 45° angle. (Wow! Who knew?)

Using the formula in **1b**, $\tan\theta = \dfrac{\frac{2}{7} - \frac{9}{5}}{1 + \frac{2}{7}\cdot\frac{9}{5}} = \dfrac{\frac{10-63}{35}}{\frac{35+18}{35}} = \dfrac{-53}{53} = -1$ so $\theta = -\frac{\pi}{4} = -45°$

5. a. Again consider $y = 2x$ and $y = 3x$. Suppose that line $y = kx$ is the angle bisector of those lines. Find the slope k. ***Hint:*** The angle between $y = 2x$ and $y = kx$ is the same as the angle between $y = kx$ and $y = 3x$. Apply the formula from problem **1b** twice, set two expressions equal, and solve for k.

Find α, the angle between kx and $2x$: $\tan\alpha = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{k-2}{1+2k}$ and β, the angle between $3x$ and kx: $\tan\beta = \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{3-k}{1+3k}$. We know $\alpha = \beta$ because kx is an angle bisector so $\dfrac{k-2}{1+2k} = \dfrac{3-k}{1+3k}$. Using our knowledge of rational equations, this can be solved to show that $k = 1 \pm \sqrt{2}$

b. Using the same method, develop a generalized formula for the slope of the angle bisector between any two lines of slopes m_1 and m_2. See *Advanced Math.* page 379 exercise 27a if you need a hint.

Starting with $\dfrac{k - m_1}{1 + km_1} = \dfrac{m_2 - k}{1 + km_2}$ and cross multiplying/combining like terms, putting the resulting quadratic $(m_1 + m_2)k^2 + 2(1 - m_1 m_2)k - (m_1 + m_2) = 0$ into the quadratic formula yields the following formula for k: $k = \dfrac{(m_1 m_2 - 1) \pm \sqrt{(1 - m_1 m_2)^2 + (m_1 + m_2)^2}}{(m_1 + m_2)}$.

6. Prove these identities. You may use any other identities we've already established in class.

These are from *Brown*, p.374/39 – 41.

a. Prove that $\cos A \cos B = \frac{1}{2}\left(\cos(A+B)+\cos(A-B)\right)$.

Hint: Start with the right side; substitute known identities for $\cos(A+B)$ and $\cos(A-B)$.

b. Derive a similar formula relating $\sin A \sin B$, $\cos(A+B)$, and $\cos(A-B)$.

c. Prove that $\sin A \cos B = \frac{1}{2}\left(\sin(A+B)+\sin(A-B)\right)$.

7. Calculate $\sin(-\frac{\pi}{12})$ two different ways as specified in parts **a** and **b**. Give exact expressions for your answers (involving $\sqrt{\ }$'s).

a. … using the fact that $-\frac{\pi}{12}=\frac{\pi}{4}-\frac{\pi}{3}$ and applying an angle difference identity.

Remember partial fraction decomposition?

b. … using the fact that $-\frac{\pi}{12}=\frac{1}{2}\cdot(-\frac{\pi}{6})$ and applying a half-angle identity.

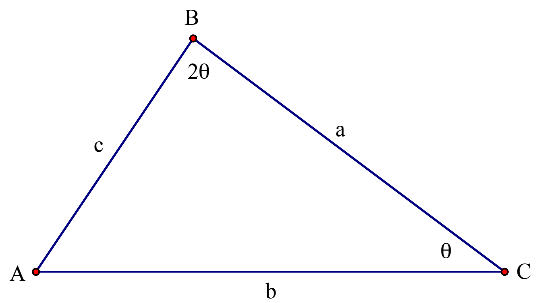
Self-explanatory, you just want to be able to do it without your calculator or notes.

c. Parts **a** and **b** should have produced different-looking expressions. But compare these three decimal approximations on your calculator: the answer to part **a**, the answer to part **b**, and the value of $\cos(-\frac{\pi}{12})$ from using the calculator's cosine operation. Check that all of these decimals are equal (make sure you're not off by a – sign).

8. Do *Advanced Math* (pages 384–385) 10.3 Written Exercises 37, 51, 52, 53.

Hint on problem 37: First step $\tan\frac{x}{2}=\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$, then use half-angle formulas, then combine the $\sqrt{\ }$'s, then multiply top-and-bottom by $(1+\cos x)$, etc.

52. a. $\dfrac{\sin\theta}{c}=\dfrac{\sin 2\theta}{b}$ and solve for b.



b.

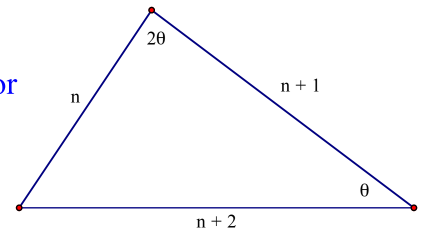
$$
\begin{aligned}
b^2 &= a^2+c^2-2ac\cdot\cos(2\theta)\\
&= a^2+c^2-2ac\left(2\cos^2(\theta)-1\right)\\
&= a^2+c^2-4ac\cdot\cos^2(\theta)+2ac\\
&= a^2+c^2+2ac-(2c\cdot\cos(\theta))(2a\cdot\cos(\theta))\\
&= a^2+c^2+2ac-(2ab\cdot\cos(\theta))\\
&= a^2+c^2+2ac+c^2-a^2-b^2\\
2b^2 &= 2c^2+2ac\\
b^2 &= c(c+a)
\end{aligned}
$$

Note the substitution of b for $2c\cos(\theta)$ in the 4th line and the use of $c^2=a^2+b^2-2ab\cos(\theta)$ to replace $-2ab\cos(\theta)$ in the 6th line.

53. a. $\frac{\sin\theta}{n} = \frac{\sin 2\theta}{n+2}$ so $\frac{\sin\theta}{n} = \frac{2\sin\theta\cos\theta}{n+2}$ and solve for cos(θ)

b. $n^2 = (n+1)^2 + (n+2)^2 - 2(n+1)(n+2)\cos(\theta)$, expland and solve for cos(θ)

c. *Set the two expressions for n equal to each other to get that n = 4.*



9. Do *Advanced Math* (pages 390–391) 10.4 Written Exercises 23, 26, 27, 28, 43, 44