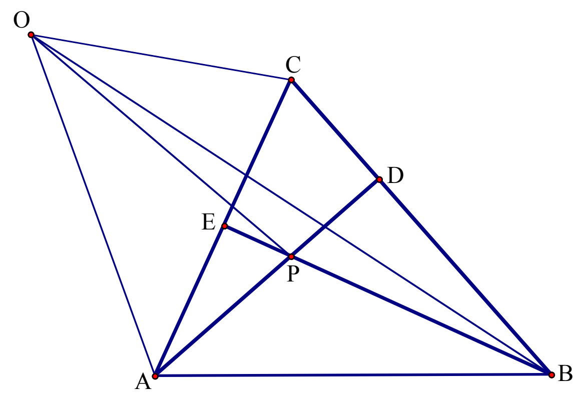
Here are the solutions to two problems that we did not have time to finish in class. They employ two different methods of using vector proofs. The first uses an external point to create reference vectors. The second uses vectors that are part of the figure already.

Proof 1: Prove the three altitudes of a triangle are concurrent.



Start with triangle ABC, altitudes AD and BE, and exterior point O. AD and BE intersect at P. If you can prove that vector CP is perpendicular to vector AB then you have shown that the altitudes are concurrent at P.

By employing point O, we can make several shared vectors, $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, and $\vec{OP}$. To put each of these vectors into component form, we will use the notation $\langle a_x, a_y\rangle$, $\langle b_x, b_y\rangle$, $\langle c_x, c_y\rangle$, and $\langle p_x, p_y\rangle$.

$\vec{AP} = -\vec{OA} + \vec{OP}$ and $\vec{CB} = -\vec{OC} + \vec{OB}$

We know $\vec{AD} \perp \vec{CB}$ so $\vec{AP} \perp \vec{CB}$ which means $\vec{AP} \cdot \vec{CB} = 0$ therefore $\left(\vec{OP} - \vec{OA}\right) \cdot \left(\vec{OB} - \vec{OC}\right) = 0$.

To simplify the equations, knowing that the x and y components will behave in the same way, we will write this as: $(p - a) \cdot (b - c) = 0$. Expanded, this takes the form $pb - pc - ab + ac = 0$. Call this equation **1**.

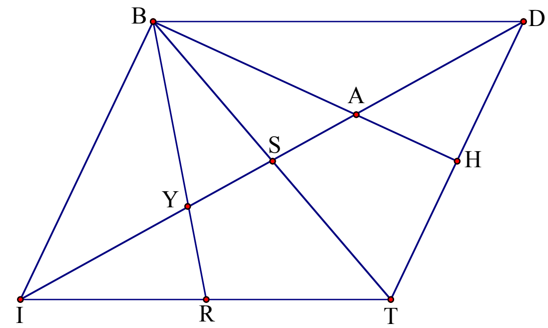
Next, $\vec{BP} = -\vec{OB} + \vec{OP}$ and $\vec{CA} = -\vec{OC} + \vec{OA}$

We know $\vec{BE} \perp \vec{CA}$ so $\vec{BP} \perp \vec{CA}$ which means $\vec{BP} \cdot \vec{CA} = 0$ therefore $\left(\vec{OP} - \vec{OB}\right) \cdot \left(\vec{OA} - \vec{OC}\right) = 0$.

We will write this as: $(p - b) \cdot (a - c) = 0$. Expanded, this takes the form $pa - pc - ab + bc = 0$. Call this equation **2**.

Subtract equation **1** from equation **2** to get: $pa - pb - ac + bc = 0$. Factor to get $(p - c) \cdot (a - b) = 0$ which can be written as $\left(\vec{OP} - \vec{OC}\right) \cdot \left(\vec{OA} - \vec{OB}\right) = 0$ so $\vec{CP} \cdot \vec{BA} = 0$. Now we can say $\vec{CP} \perp \vec{BA}$ which puts $\vec{CP}$ on the altitude so the three altitudes are concurrent.

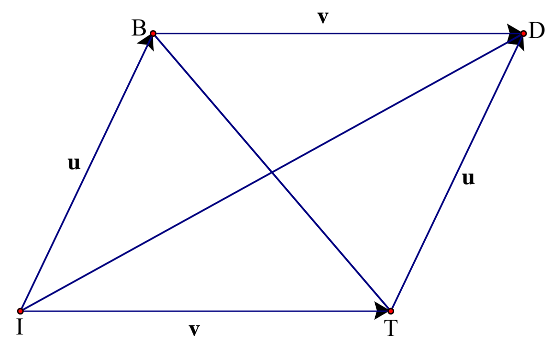
Proof 2: Prove the vectors from one vertex of a parallelogram to the midpoints of the opposite sides trisect the diagonal.



Point names have been added to reference ease. The goal is to prove that BR and BH trisect ID.

① **To** help us, we need to remember that we have already proved that $\vec{IS} = \frac{1}{2}\vec{IB} + \frac{1}{2}\vec{IT}$ and $\vec{DS} = \frac{1}{2}\vec{DB} + \frac{1}{2}\vec{DT}$ (in class we proved $\vec{OC} = x\vec{OA} + y\vec{OB}$ when C is between A and B on a line).

② We also proved that the centroid is $\frac{2}{3}$ of the way from a vertex to the opposite midpoint in a triangle.



Sides IB and TD are vector **u**. Sides BD and IT are vector **v**. Diagonal BT is **u – v** and diagonal ID is **u + v.**

Using ① we know that IS is $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}$ and that DS is $-\frac{1}{2}\mathbf{u} - \frac{1}{2}\mathbf{v}$.

Using ② we know that IY is $\frac{2}{3}$IS so $\frac{2}{3}\left(\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{v}\right) = \left(\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v}\right) = \frac{1}{3}(\mathbf{u} + \mathbf{v})$.

A similar argument can be used to show that DA is $-\frac{1}{3}(\mathbf{u} + \mathbf{v})$ which leaves a third of ID for YA.