Summary of the Properties of Determinants

Notation: The determinant of matrix T can be denoted by either $\det T$ or $|T|$. (Caution: $|T|$ does not mean "the absolute value of T." $|T|$ can be positive, negative, or zero.)

The 2 by 2 Case: If $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then:	**The 3 by 3 Case:** If $T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then:
1. Calculation: $\det T = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.	**1.** Calculation: $\det T = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ $= a_1(b_2c_3 - c_2b_3) - b_1(a_2c_3 - c_2a_3) + c_1(a_2b_3 - b_2a_3)$
2. The matrix T maps the unit square to the parallelogram formed by the vectors $\langle a,c \rangle$ and $\langle b,d \rangle$. (Remember that squares and rectangles are parallelograms.)	**2.** The matrix T maps the unit cube to the parallelepiped formed by the vectors $\langle a_1,a_2,a_3 \rangle$, $\langle b_1,b_2,b_3 \rangle$, and $\langle c_1,c_2,c_3 \rangle$.
3. The absolute value of the determinant of T is the *area stretch factor* for the transformation. In other words, when the unit square goes through T, the area of the resulting parallelogram is given by $\lvert\det T\rvert$.	**3.** The absolute value of the determinant of T is the *volume stretch factor* for the transformation. In other words, when the unit cube goes through T, the volume of the resulting parallelepiped is given by $\lvert\det T\rvert$.
4. If $\det T$ is positive, the transformation is *orientation preserving*. If $\det T$ is negative, the transformation is *orientation reversing*.	**4.** If $\det T$ is positive, the transformation is *orientation preserving*. If $\det T$ is negative, the transformation is *orientation reversing*.

The 2 by 2 Case: If $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then:	**The 3 by 3 Case:** If $T = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, then:
5. If $\det T \neq 0$, then all of the following are true: **a.** T^{-1} exists. **b.** The system of equations $\begin{pmatrix} ax + by = e \\ cx + dy = f \end{pmatrix}$ has a unique solution. **c.** The lines intersect at a single point. **d.** The mapping T goes from the xy-plane to the xy-plane. **e.** The transformation T is a one-to-one mapping.	**5.** If $\det T \neq 0$, then all of the following are true: **a.** T^{-1} exists. **b.** The system of equations $\begin{pmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{pmatrix}$ has a unique solution. **c.** The planes intersect at a single point. **d.** The mapping T goes from 3-space to 3-space. **e.** The transformation T is a one to one mapping.
6. If $\det T = 0$, then all of the following are true: **a.** T^{-1} does not exist. **b.** The system of equations $\begin{pmatrix} ax + by = e \\ cx + dy = f \end{pmatrix}$ does not have a unique solution. **c.** The lines do not intersect at a single point. (If the two lines are identical, then every point on the line is a solution. If the two lines are parallel, there is no solution. **d.** The mapping T results in the loss of at least one dimension. In other words, the entire xy-plane is mapped to a single line (the loss of one dimension) or, if T is the zero matrix, the entire xy-plane is mapped to the origin (the loss of both dimensions). **e.** The transformation T is a not a one-to-one mapping.	**6.** If $\det T = 0$, then all of the following are true: **a.** T^{-1} does not exist. **b.** The system of equations $\begin{pmatrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{pmatrix}$ does not have a unique solution. **c.** The planes do not intersect at a single point. The intersection could be a plane (if all three equations are identical), a line, or a no intersection. Note: The "no intersection" case does not require any of the planes to be parallel. (Can you picture how this would occur?) Algebraically, the no intersection case happens when one of the planes is a *linear combination* of the other two. **d.** The mapping T results in the loss of a dimension. In other words, all of 3-space is mapped to a plane (the loss of 1 dimension), to a line (the loss of 2 dimensions), or to the origin (the loss of all dimensions). **e.** The transformation T is not a one-to-one mapping.