Honors Advanced Math — Name:________________________

Review/Extension 3D Vector Problems — Block: E or G   14 April 2014

Three-dimensional review and extension problems

An introduction to problem **1***:*

I did Problem **1** in class but now you will derive the justification for the Cartesian equation of a plane. It involves developing a formula for the plane with a specified perpendicular vector and passing through a specified point. This formula, proved in **1b**, is useful in its own right. You can think of it as the point-vector equation form for a plane in 3-D space.

1. Consider the plane that is perpendicular to vector $<a, b, c>$ which has (x_0, y_0, z_0) as one of its points. Let (x_0, y_0, z_0) and (x, y, z) stand for the coordinates of two points in the plane.

- **a.** Explain why vectors $<a, b, c>$ and $<x - x_0, y - y_0, z - z_0>$ are perpendicular.
- **b.** Prove that $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
- **c.** Rewrite the equation proved in part **b** to have the form $ax + by + cz =$ [some expression].
- **d.** Now explain why any plane $ax + by + cz = d$ must be perpendicular to $<a, b, c>$.

2. Repeat problem **1** in 2-dimensional space. Consider the line that is perpendicular to $<a, b>$ and has (x_0, y_0) as one of its points. Take steps analogous to parts **a**, **b**, **c**, and **d**. Then summarize what you have proved, and how it relates to what you previously knew about 2-D lines (be sure to include a comparison to standard and point-slope form).

3. Consider the motion along a line in 3-dimensional space described by these parametric equations: $x = 4 + 3t$, $y = 5 - t$, $z = 6 - 2t$.

- **a.** Combine the three parametric equations into a single equation giving (x, y, z) as a function of t. Then combine the three equations into a single vector equation that involves an initial point and a velocity vector.
- **b.** What is the speed of the moving object? (That is, how fast is it going, per unit of time?) The answer should be a number not a vector.
- **c.** Write a new vector equation for an object that is moving along the same line as the original object but twice as fast.

4. Remember the *x*-*y*-*z* description of a line in 3-D space, in a form we have not focused on:

$$\frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}.$$

- **a.** Combine the three equations from part **a** into a single equation for the line, involving a point and a vector.
- **b.** If you regard the equations from parts **a** and **b** as giving the position of a moving object at time t, find the speed of the moving object.
- **c.** Write an equation for the plane that is perpendicular to the given line at (2, 1, 2).

5. Vector $\mathbf{v} = \langle 2, 3, 4 \rangle$ sits on plane $\boldsymbol{M}$: $x - 4y + 2z = 8$ at the point (8, 1, 2).

- **a.** Find the angle between vector $\mathbf{v}$ and plane $\boldsymbol{M}$.
- **b.** Find the distance between the end of the vector and the plane.
- **c.** Project the vector on the plane.
- **d.** Project the vector on the vector perpendicular to the plane.

6. Consider a parallelogram in 3-dimensional space, whose sides are described by the vectors $\mathbf{v} = \langle 2, 3, 4 \rangle$ and $\mathbf{w} = \langle -1, 5, -2 \rangle$. The goal of this problem is to find the parallelogram's area, two different ways (and of course the answers should agree).

- **a.** Find the angle between vectors **v** and **w**, then find the parallelogram's area using the trigonometric SAS area formula for triangles.
- **b.** Find the parallelogram's area using the cross-product of the vectors.

7. Now consider a parallelepiped (3-D "box" with parallelogram faces) formed by vectors $\mathbf{u} = \langle -3, 1, 0 \rangle$, $\mathbf{v} = \langle 2, 3, 4 \rangle$ and $\mathbf{w} = \langle -1, 5, -2 \rangle$. Note that **v** and **w** are the same as in the previous problem. Find the volume of this parallelepiped. ***Hint:*** Recall that any prism's volume equals (*area of the prism's base*)·(*height of the prism*). Suppose you regard the face formed by **v** and **w** as being the base of the prism. Find the height of the prism using **u** and the techniques from problem **3**.

8. Consider points A = (2, 3, 4), B = (5, 7, –8), and C = (–10, 8, 4).

- **a.** Find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in component form.
- **b.** Find a vector that is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
- **c.** Find the *xyz*-equation of the plane containing A, B, and C.
- **d.** If point D = (3, 5, 1), are points A, B, C, and D coplanar? Justify your answer.

9. Consider point X = (–1, 4, 3) and plane *P* whose equation is $2x - y + z = 4$.

- **a.** Identify a vector that is perpendicular to plane *P*.
- **b.** Let *l* be the line that goes through point X and is perpendicular to plane *P*. Write either a vector equation or a set of three parametric equations for line *l*.
- **c.** Find the coordinates of the point where line *l* and plane *P* intersect.
- **d.** Find the distance from point X to plane *P*. *Hint:* Distance between a point and a plane means the shortest distance, and this shortest distance is along the perpendicular line *l*.

10. Consider point X = (–1, 4, 3) and line l described by $(x, y, z) = (3, 0, -4) + t\langle 1, 4, 2\rangle$.

a. Find an equation for plane P that contains point X and is perpendicular to line l.

b. Find the coordinates of the point where P and l intersect perpendicularly.

b. Now find the distance from point X to line l.

11. Consider two planes whose equations are $2x - 5y + 3z = 12$ and $3x + 4y - 3z = 6$.

a. Find the measure of the angle between the planes.

b. Find the equation of the line of intersection of the planes. ***Hint:*** Think about the angle the intersection line makes with the vectors perpendicular to the planes.

12. a. Find an equation for the plane determined by the points (1, 2, 3), (4, 2, 0), and (–1, –2, 5).

b. Do the points (5, 0, 2), (3, –1, 4), and (–1, –3, 8) determine a plane? Justify your answer.

13. The planes $x - 3y + 4z = 6$ and $x - 3y + 4z = 10$ are parallel (as proved in problem **1d**). Find the distance between these planes. *Hint:* Choose any line perpendicular to both planes, find the points where the line intersects the planes, then find the distance between those points.

14. Consider the vectors $\mathbf{v} = \langle \cos a, \sin a, 0\rangle$ and $\mathbf{w} = \langle \cos b, \sin b, 0\rangle$, where $a > b > 0$.

a. Find the dot product $\mathbf{v} \cdot \mathbf{w}$, and use the result to prove the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

b. Find the cross product $\mathbf{v} \times \mathbf{w}$, and use the result to prove the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

15. Prove that, for any vectors **v** and **w**, if $|\mathbf{v}| = |\mathbf{w}|$ then $\mathbf{v} + \mathbf{w}$ is perpendicular to $\mathbf{v} - \mathbf{w}$.

16. ABCDEFGH is a regular octagon. The vector $\overrightarrow{AB} = \mathbf{p}$, the vector $\overrightarrow{BC} = \mathbf{q}$. Express the vector $\overrightarrow{AH}$ in terms of **p** and **q**.

ANSWERS

1. See your notes or Brown p. 453.

2. I expect great things from you.

3. **a.** $\dfrac{x-4}{3} = 5 - y = \dfrac{z-6}{-2}$ or $(x, y, z) = (4, 5, 6) + t\langle 3, -1, -2\rangle$.

b. speed = $|\langle 3, -1, -2\rangle| = \sqrt{14}$

c. $(x, y, z) = (4, 5, 6) + t\langle 6, -2, -4\rangle$.

4. **a.** $x = -3t + 2$, $y = 4t + 1$, $z = -t + 2$ so $(x, y, z) = (2, 1, 2) + t\langle -3, 4, -1\rangle$.

b. speed = magnitude of $\langle -3, 4, -1\rangle$, which is $\sqrt{26}$

c. Write $-3x + 4y - z = d$, then substitute (2, 1, 2) for (x, y, z). Answer: $-3x + 4y - z = -4$.

5. **a.** 4.649°

b. 0.436

c. length = 5.367 but that is not enough so use the line perpendicular to the plane through the point at the end of the vector and find $t = 2/21$ thus the projection is

$$\mathbf{v} + \frac{2}{21}\langle 1,\ -4,\ 2\rangle = \left\langle \frac{44}{21},\ \frac{71}{21},\ \frac{88}{21} \right\rangle$$

d. $\langle -0.095, 0.381, -0.191\rangle$

6. **a.** $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \dfrac{5}{\sqrt{29}\sqrt{30}}$, whose inverse cosine is $\theta \approx 80.24°$.

Area = $|\mathbf{v}|\,|\mathbf{w}|\sin\theta \approx \sqrt{29}\sqrt{30}\sin(80.24°) \approx 29.069$.

b. $|\mathbf{v} \times \mathbf{w}| = |\langle 26, 0, 13\rangle| = 13\sqrt{5} = 29.069$

7. Volume = Area of the Base * Height = $|\mathbf{v} \times \mathbf{w}| * |\mathbf{u}|\cos\theta$ where θ is the angle between vector **u** and the plane formed by *v* and **w**. The volume is 78.

8. **a.** **AB** = $\langle 3, 4, -12\rangle$ and **AC** = $\langle -12, 5, 0\rangle$

b. $\langle 60, 144, 63\rangle = 3\langle 20, 48, 21\rangle$

c. $20x + 48y + 21z = 268$

d. $20(3) + 48(5) + 21(1) = 321 \neq 268$ so <u>not</u> coplanar

9. **a.** $\langle 2, -1, 1\rangle$

b. $(x, y, z) = (-1, 4, 3) + t\langle 2, -1, 1\rangle$, then split into separate equations for x, y, and z

c. Substitute the three parametric equations into $2x - y + z = 4$ to get $2(-1 + 2t) - (4 - t) + (3 + t) = 4$. Solve for t to get $t = \frac{7}{6}$. Then use the parametric equations to get x, y, and z. Answer: $(\frac{8}{6}, \frac{17}{6}, \frac{25}{6})$.

d. Calculate distance between $(-1, 4, 3)$ and $(\frac{8}{6}, \frac{17}{6}, \frac{25}{6})$
or calculate $\frac{7}{6}$ times the magnitude of $\langle 2, -1, 1\rangle$. Answer: $\frac{7\sqrt{6}}{6}$.

10. a. $x + 4y + 2z = 21$.

b. Turn the line's equation into three parametric equations, then substitute into the plane's equation. Get $(3 + t) + 4(4t) + 2(-4 + 2t) = 21$. Solve for t, get $t = \frac{26}{21}$. Substitute $t = \frac{26}{21}$ into the parametric equations to get $(x, y, z) = (\frac{89}{21}, \frac{96}{21}, -\frac{32}{21})$.

c. Using the distance formula, find the distance between (–1, 4, 3) and $(\frac{89}{21}, \frac{96}{21}, -\frac{32}{21})$.
$d =$ 26.519

11. a. angle between <2, –5, 3> and <3, 4, –3> is 50.217°

b. the vector perpendicular to <2, –5, 3> and <3, 4, –3> is in both planes so <3, 15, 23> is the vector for the line. Find a point on both planes by assuming *x, y,* or *z* is some value and solve for the other two (ex. if $x = 0$ then (0, –12, 34) is on both planes) so the equation is $(x, y, z) = (0, -12, 34) + t$ <3, 15, 23>

12. a. Call the points $A(1, 2, 3)$, $B(4, 2, 0)$, and $C(-1, -2, 5)$. ***AB*** = <3, 0, –3> and ***AC*** = <–2, –4, 2> so a vector equation of the plane is $(x, y, z) = (1, 2, 3) + t$<3, 0, –3> + s<–2, –4, 2>.

b. Call the points $D(5, 0, 2)$, $E(3, -1, 4)$, and $F(-1, -3, 8)$. ***DE*** = <–2, –1, 2> and ***DF*** = <–6, –3, 6> which is 3<–2, –1, 2> so they are collinear points.

13. Follow the method described in the hint. The easiest line to choose is $(x, y, z) = (0, 0, 0) + t$< 1, –3, 4>. The final answer is $\sqrt{\frac{8}{13}} = \frac{2\sqrt{26}}{13}$ (or an equivalent expression).

14. and 15. I can't type the proofs right now.

16. The easiest way to try this is to put the octagon on a coordinate grid with point A at the origin. Make each side of length one so that point B is at (0, 1) (which means that **p** = <0, 1>). Once you have figured out the vector **q** and vector ***AH*** then you can make the linear combination of **p** and **q** to determine that $\boldsymbol{AH} = \mathbf{q} - \sqrt{2}\,\mathbf{p}$.